Finite Horizon Analysis of Markov Chains with the Murφ Verifier*

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Abstract. In this paper we present an explicit disk based verification algorithm for Probabilistic Systems defining *discrete time/finite state* Markov Chains. Given a Markov Chain and an integer k (horizon), our algorithm checks whether the probability of reaching an error state in at most k steps is below a given threshold.
We present an implementation of our algorithm within a suitable extension of the Murφ verifier. We call the resulting probabilistic model checker FHP-Murφ (*Finite Horizon Probabilistic* Murφ).
We present experimental results comparing FHP-Murφ with (a finite horizon subset of) PRISM, a state-of-the-art symbolic model checker for Markov Chains. Our experimental results show that FHP-Murφ can handle systems that are out of reach for PRISM, namely those involving arithmetic operations on the state variables (e.g. hybrid systems).

1 Introduction

Model checking techniques [5,11,16,15,21,28] are widely used to verify correctness of digital hardware, embedded software and protocols by modeling such systems as *Nondeterministic Finite State Systems* (NFSSs).

However, there are many reactive systems that exhibit uncertainty in their behaviour, i.e. which are stochastic systems. Examples of such systems are: fault tolerant systems, randomized distributed protocols and communication protocols. Typically stochastic systems cannot be conveniently modeled using NFSSs. However, they can often be modeled by *Markov Chains* [2,12]. Roughly speaking, a Markov Chain can be seen as an automaton labelled with (outgoing) probabilities on its transitions.

For stochastic systems correctness can only be stated using a probabilistic approach, e.g. using a *Probabilistic Logic* (e.g. [32,8,13]). This motivates the development of *Probabilistic Model Checkers* [9,1,17], i.e. of model checking algorithms and tools whose goal is to automatically verify (probabilistic) properties

* This research has been partially supported by MURST projects: MEFISTO and SAHARA.

D. Geist and E. Tronci (Eds.): CHARME 2003, LNCS 2860, pp. 394–409, 2003.

of stochastic systems (typically Markov Chains). For example, a probabilistic model checker may automatically verify a system property like "the probability that a message is not delivered after 0.1 seconds is less than 0.80".

Many methods have been proposed for probabilistic model checking, e.g. [10, 3,8,13,14,19,24,27,32].

To the best of our knowledge, currently, the state-of-the-art probabilistic model checker is PRISM [25,1,18]. PRISM overcomes the limitations due to the use of linear algebra packages in Markov Chain analysis by using *Multi Terminal Binary Decision Diagrams* (MTBDDs) [6], a generalization of *Ordered Binary Decision Diagrams* (OBDDs) [4] allowing real numbers in the interval $[0, 1]$ on terminal nodes. More precisely, PRISM can carry out the required Markov Chain analysis using a matrix based approach (based on linear algebra packages), a symbolic approach (based on the CUDD package [7]) as well as a hybrid approach. The user can choose the best approach for the problem at hand.

Here we are mainly interested in automatic analysis of *discrete time/finite state* Markov Chains modeling *Discrete Time Hybrid Systems*. Such Markov Chains can in principle be analyzed using PRISM. However, our experience is that, using PRISM on our systems, quite soon we run into a *state explosion* problem, i.e. we run out of memory because of the huge OBDDs built during the model checking process. This is due to the fact that hybrid systems dynamics typically entails many arithmetical operations on the state variables. This makes life very hard for OBDDs, thus making usage of a symbolic probabilistic model checker (e.g. like PRISM) on such systems rather problematic.

Indeed our experience shows that *Explicit Model Checking* can outperform *Symbolic Model Checking* in automatic analysis of *Hybrid Control Systems* [22]. This suggested us to explore the possibility of devising an explicit disk based algorithm for automatic *Finite Horizon* safety analysis of Markov Chains. In this paper we present our algorithm as well as experimental results showing its effectiveness. Our results can be summarized as follows.

- We present (Sections 3, 4) an *explicit* algorithm for automatic verification of discrete time/finite state Markov Chains. Given a Markov Chain $\mathcal{M}$, our algorithm checks wheter the probability of reaching a given state s within k steps is less than a given bound p. Our algorithm is *disk based*, thus, because of the large size of modern hard disks, state explosion is hardly a problem for us. Computation time instead is our bottleneck. Our algorithm can trade RAM memory with computation time, i.e. the more RAM available the faster our computation. To the best of our knowledge, this is the first time that such a disk based algorithm for probabilistic model checking is proposed.
- We present (Sections 5) an implementation of our algorithm within the Murφ [21] verifier. We call the resulting probabilistic model checker FHP-Murφ (*Finite Horizon Probabilistic* Murφ).
- We present (Section 6.1) experimental results comparing FHP-Murφ with PRISM on two suitably modified versions of the dining philosophers protocol included in the PRISM distribution. Our experimental results show that FHP-Murφ can handle systems that are out of reach for PRISM. However,

as long as PRISM does not hit state explosion, PRISM is faster than FHP-Murφ (as to be expected).
Note however that PRISM can handle more general models than FHP-Murφ, and can verify more general properties (namely all PCTL [13] properties) than FHP-Murφ. In fact, FHP-Murφ can only verify finite horizon safety properties for Markov Chains, a subclass (although an important one) of the verification tasks that PRISM can handle.

– We present (Section 6.2) experimental results on using FHP-Murφ for a probabilistic analysis of a "real world" hybrid system, namely the Turbogas Control System of the Co-generative power plant described in [22]. Because of the arithmetic operations involved in the definition of system dynamics, this hybrid system is out of reach for OBDDs (and thus for PRISM), whereas FHP-Murφ can complete (finite horizon) verification within reasonable time.

2 Basic Notation

Let S be a finite set. We regard functions from S to the real interval [0, 1] and functions from $S \times S$ to [0, 1] as row vectors and as matrices, respectively. If $\mathbf{x}$ is a vector and $s \in S$ we also write $\mathbf{x}_s$ or $(\mathbf{x})_s$ for $\mathbf{x}(s)$. If $\mathbf{P}$ is a matrix and $s, t \in S$ we also write $\mathbf{P}_{s,t}$ or $(\mathbf{P})_{s,t}$ for $\mathbf{P}(s,t)$. On vectors and matrices we use the standard matrix operations. Namely: $\mathbf{xP}$ is the row vector $\mathbf{y}$ s.t. $\mathbf{y}_s = \sum_{j \in S} \mathbf{x}_j \mathbf{P}_{j,s}$ and $\mathbf{AB}$ is the matrix $\mathbf{C}$ s.t. $\mathbf{C}_{s,t} = \sum_{j \in S} \mathbf{A}_{s,j} \mathbf{B}_{j,t}$. We define $\mathbf{A}^n$ in the usual way, i.e.: $\mathbf{A}^0 = \mathbf{I}$, $\mathbf{A}^{n+1} = \mathbf{A}^n \mathbf{A}$, where $\mathbf{I}$ (*the identity matrix*) is the matrix defined as follows: $\mathbf{I}(s,j) = \textbf{if } (s = j) \textbf{ then } 1 \textbf{ else } 0$. We denote with $\mathcal{B}$ the set $\{0, 1\}$ of boolean values. As usual 0 stands for *false* and 1 stands for *true*.

We give some basic definitions on Markov Chains. For further details see, e.g. [2]. A *distribution* on S is a function $\mathbf{x} : S \to [0, 1]$ s.t. $\sum_{i \in S} \mathbf{x}(i) = 1$. Thus a distribution on S can be regarded as a $|S|$-dimensional row vector $\mathbf{x}$. A distribution $\mathbf{x}$ represents *state* $j \in S$ iff $\mathbf{x}(j) = 1$ (thus $\mathbf{x}(i) = 0$ when $i \neq j$). If distribution $\mathbf{x}$ represents $s \in S$, by abuse of language we also write $\mathbf{x} \in S$ to mean that distribution $\mathbf{x}$ represents a state and we use $\mathbf{x}$ in place of the element of S represented by $\mathbf{x}$. In the following we often represent states using distributions. This allows us to use matrix notation to define our computations.

Definition 1. *1. A* Discrete Time Markov Chain (*just* Markov Chain *in the following*) *is a triple $\mathcal{M} = (S, \mathbf{P}, q)$ where: S is a finite set (of* states*), $q \in S$ and $\mathbf{P} : S \times S \to [0, 1]$ is a transition matrix, i.e. for all $s \in S$, $\sum_{t \in S} \mathbf{P}(s,t) = 1$. (We included the* initial state *q in the Markov Chain definition since in our context this will often shorten our notation.)*

2. An execution sequence (*or* path) *in the Markov Chain $\mathcal{M} = (S, \mathbf{P}, q)$ is a nonempty (finite or infinite) sequence $\pi = s_0 s_1 s_2 \ldots$ where s_i are states and $\mathbf{P}(s_i, s_{i+1}) > 0$, $i = 0, 1, \ldots$. If $\pi = s_0 s_1 s_2 \ldots$ we write $\pi(k)$ for s_k. The* length *of a finite path $\pi = s_0 s_1 s_2 \ldots s_k$ is k (number of transitions), whereas the length of an infinite path is ω. We denote with $|\pi|$ the length of π. We*

denote with $Path(\mathcal{M}, s)$ *the set of infinite paths* π *in* $\mathcal{M}$ *s.t.* $\pi(0) = s$. *If* $\mathcal{M} = (S, \mathbf{P}, q)$ *we write also* $Path(\mathcal{M})$ *for* $Path(\mathcal{M}, q)$.

3. *For* $s \in S$ *we denote with* $\sum(s)$ *the smallest* σ*-algebra on* $Path(\mathcal{M}, s)$ *which, for any finite path* ρ *starting at* s, *contains the basic cylinders* $\{ \pi \in Path(\mathcal{M}, s) \mid \rho \text{ is a prefix of } \pi \}$. *The probability measure* Pr *on* $\sum(s)$ *is the unique measure with* $Pr(\{\pi \in Path(\mathcal{M}, s) | \rho \text{ is a prefix of } \pi\}) = Pr(\rho) = \prod_{i=0}^{k-1} \mathbf{P}(\rho(i),\ \rho(i+1)) = \mathbf{P}(\rho(0), \rho(1))\mathbf{P}(\rho(1), \rho(2)) \cdots \mathbf{P}(\rho(k-1), \rho(k))$, *where* $k = |\rho|$.

E.g. given distribution $\mathbf{x}$, the distribution $\mathbf{y}$ obtained by one execution step of Markov Chain $\mathcal{M} = (S, \mathbf{P}, q)$ is computed as: $\mathbf{y} = \mathbf{xP}$. In particular if $\mathbf{y} = \mathbf{xP}$ and $\mathbf{x}(s) = 1$ we have that $\forall t[\mathbf{y}(t) = (\mathbf{P})_{s,t}]$.

3 Finite Horizon Safety Verification of Markov Chains

Given a Markov Chain, we want to compute the probability that a path of length k starting from a given initial state q reaches a state s satisfying a given boolean formula ϕ (i.e. $\phi(s) = 1$). If ϕ models an *error condition* the above computation allows us to compute the probability of reaching an error condition in at most k transitions.

Problem 1. Let $\mathcal{M} = (S, \mathbf{P}, q)$ be a Markov Chain, $k \in \mathbb{N}$, and ϕ be a boolean function on S. We want to compute: $P(\mathcal{M}, k, \phi) = Pr((\exists i \leq k\ \phi(\pi(i))) \mid \pi \in \text{Path}(\mathcal{M}))$ That is, we want to compute the probability of reaching a state satisfying ϕ in *at most* k steps in Markov Chain $\mathcal{M}$ (starting from $\mathcal{M}$ initial state q).

Definition 2. *Let* $\mathcal{M} = (S, \mathbf{P}, q)$ *be a Markov Chain and let* ϕ *be a boolean function on* S, *i.e.* $\phi : S \to \mathcal{B}$. *We define Markov Chain* $\mathcal{M}_\phi$ *as follows.*

$\mathcal{M}_\phi = (S, \mathbf{P}_\phi, q)$, *where for all* $s, t \in S$, $\mathbf{P}_\phi(s,t) = \begin{cases} \mathbf{P}(s,t) & \text{if } \neg\phi(s) \\ 1 & \text{if } \phi(s) \wedge (s = t) \\ 0 & \text{if } \phi(s) \wedge (s \neq t) \end{cases}$

In other words, Markov Chain $(S, \mathbf{P}_\phi, q)$ is obtained from $(S, \mathbf{P}, q)$ by removing all outgoing edges from any state s satisfying ϕ (error state) and replacing such outgoing edges with just one edge leading back to s. Thus, once an error state is entered there is no way to leave it. This, in turn, means that for $(S, \mathbf{P}_\phi, q)$ the probability of reaching in *exactly* k steps a state satisfying ϕ is exactly the same as the probability of reaching in *at most* k steps a state satisfying ϕ. Note that according to item 1 of Definition 1 $(S,\ \mathbf{P}_\phi,\ q)$ is indeed a Markov Chain.

From the above considerations follow that $P(\mathcal{M}, k, \phi)$ can be computed from $\mathbf{P}_\phi$ as shown in Proposition 1. Essentially Proposition 1 is a specialization to our finite horizon case of known results on PCTL Model Checking of Markov Chains (e.g. [13,1]).

Proposition 1. *Let* $\mathcal{M} = (S, \mathbf{P}, \mathbf{q})$, *and let* ϕ *be a boolean function on* S. *Then* $P(\mathcal{M}, k, \phi) = Pr((\exists i \leq k\ \phi(\pi(i))) \mid \pi \in \text{Path}(\mathcal{M})) = \sum_{s:\phi(s)} (\mathbf{q}{\mathbf{P}_\phi}^k)_s$

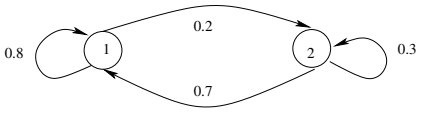

Fig. 1. A Markov Chain

Let ϕ be defined as follows: $\phi(s) = (s = 2)$, i.e. only state 2 satisfies ϕ. Then $\mathbf{P}_\phi = \begin{bmatrix} 0.8 & 0.2 \\ 0.0 & 1.0 \end{bmatrix}$. From Theor. 1 we have: $P(\mathcal{M}, 1, \phi) = 0.2$; $P(\mathcal{M}, 2, \phi) = 0.36$; $P(\mathcal{M}, 3, \phi) = 0.488$.

Example 1. Consider Markov Chain $\mathcal{M} = (S, \mathbf{P}, \mathbf{q})$ with $S = \{1, 2\}$, $\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}$ and $\mathbf{q} = [1\ 0]$ (i.e. distribution $\mathbf{q}$ denotes state 1). The usual automata-like representation for $\mathcal{M}$ is given in Fig. 1.

4 Probabilistic Finite State Systems

The Markov Chain Definition in Definition 1 is appropriate to study mathematical properties of Markov Chains. However Markov Chains arising from probabilistic concurrent systems are usually defined using a suitable programming language rather than a stochastic matrix. As a matter of fact the (huge) size of the stochastic matrix of concurrent systems is one of the main obstructions to overcome in probabilistic model checking.

Thus a Markov Chain is presented to a model checker by defining (using a suitable programming language) a *next state* function that returns the needed information about the immediate successors of a given state. The following definition formalizes this notion.

Definition 3. *A* Probabilistic Finite State System *(PFSS) $\mathcal{S}$ is a 3-tuple $(S, q, \texttt{next})$, where: S is a finite set (of states), $q \in S$ and* `next` *is a function taking a state s as argument and returning a set* $\texttt{next}(s)$ *of pairs (t, p) s.t. $\sum_{(t,p)\in\texttt{next}(s)} p = 1$.*

To a PFSS we can associate a Markov Chain in a unique way.

Definition 4. *1. Let $\mathcal{S} = (S, q, \texttt{next})$ be a PFSS. The Markov Chain $\mathcal{S}^{mc} = (S, \mathbf{P}, q)$ associated to $\mathcal{S}$ is defined as follows:* $\mathbf{P}(s,t) = \begin{cases} p & \text{if } (t,p) \in \texttt{next}(s) \\ 0 & \textit{otherwise} \end{cases}$

2. Given $k \in \mathbb{N}$ and a boolean function ϕ on S we write $P(\mathcal{S}, k, \phi)$ for $P(\mathcal{S}^{mc}, k, \phi)$ as defined in Problem 1. Thus Problem 1 for PFSSs becomes: given a PFSS $\mathcal{S}$ compute $P(\mathcal{S}, k, \phi)$.

Given a PFSS $\mathcal{S}$ we want to compute $P(\mathcal{S}, k, \phi)$ without generating the transition matrix for Markov Chain $\mathcal{S}^{mc}$. Using Proposition 1 this can be done as shown in Proposition 2.

Proposition 2. *Let $\mathcal{S} = (S, \mathbf{q}, \texttt{next})$ be a PFSS, $k \in \mathbb{N}$ and ϕ be a boolean function ϕ on S. Then $P(\mathcal{S}, k, \phi)$ can be computed as shown in Fig. 2.*

```
//For i = 1,...k+1, Q(i) is a queue of state-probability pairs (s,p);
P((S,q,next),k,φ) { i = 1; r = 0; enqueue(Q(i),(q,1));
forall i = 1...k do { while (Q(i) is nonempty) { (s,p) = dequeue(Q(i));
                    forall (t,a) ∈ next(s) do { if φ(t) { r = r + p*a; }
                                              else enqueue(Q(i+1),(t,p*a)); } } } return(r); }
```

Fig. 2. Computation of $P(\mathcal{S}, k, \phi)$.

Proof. (Sketch). *Let* $\mathcal{M} = \mathcal{S}^{mc} = (S, \mathbf{P}, \mathbf{q})$. *Consider the following sequence of distributions:* $\mathbf{y}^{(0)} = \mathbf{q}$, $\mathbf{y}^{(i+1)} = \mathbf{y}^{(i)}\mathbf{P}_\phi$ *for* $i = 0, \ldots k$. *From Proposition 1 we have that* $\sum_{s:\phi(s)} \mathbf{y}^{(k+1)}(s) = P(\mathcal{S}^{mc}, k, \phi)$. *Moreover, from Fig. 2 we have that for all* $i = 1, \ldots k+1$, *and for all* $s \in S'$, $\mathbf{y}^{(i)}(s) \neq 0$ *iff* $(s, \mathbf{y}^{(i)}(s)) \in Q(i)$. *Note that states* s *s.t.* $\phi(s) = 1$ *are not enqueued. In fact, by Definition 2, the only state reachable form such a state* s *is* s *itself. Thus, from Definition 2 of* $\mathbf{P}_\phi$, *we have that the value* r *returned by* $P((S, q, \mathtt{next}), k, \phi)$ *in Fig. 2 is exactly* $P(\mathcal{S}^{mc}, k, \phi)$.

Remark 1. Given a PFSS $\mathcal{S} = (S, q, \mathtt{next})$, $k \in \mathbb{N}$, a boolean function ϕ on S and a probability threshold p, in Section 5, exploiting Proposition 2, we will present an efficient disk based algorithm to check if it holds that $P(\mathcal{S}, k, \phi) < p$. In other words, our algorithm checks validity of a *Finite Horizon Probabilistic* (FHP) *Safety Property*. FHP safety properties are a very important class of properties. This motivates our disk based algorithm.

Of course a FHP safety property can be easily defined with a PCTL [13] formula, namely $P_{<p}[\text{true } U^{\leq k}\phi]$. Thus also the probabilistic model checker PRISM [25] can be used to verify FHP safety properties.

Note however that PRISM can handle *all* PCTL formulas, whereas our algorithm can only handle FHP safety properties. In particular PRISM can verify properties like $P_{<p}[\text{true } U\phi]$ (*the probability of reaching a state satisfying* ϕ *is less than* p). Such *unbounded horizon* properties cannot be handled with our algorithm.

5 Analysing Probabilistic Systems with the Murφ Verifier

Building on the computation scheme in Fig. 2, in the following we describe an efficient disk based algorithm to verify FHP-safety properties, as well as an implementation of such an algorithm within the Murφ verifier. We call the resulting tool FHP-Murφ (*Finite Horizon Probabilistic* Murφ).

5.1 Functions and Data Structures

FHP-Murφ input defines a PFSS $\mathcal{S} = (S, q, \mathtt{next})$ to which we will refer in the sequel. The FHP-Murφ keyword `startstate` defines $\mathcal{S}$ initial state q. Indeed, Murφ can have a set of initial states, however, w.l.o.g. in the following we assume

```
int k; /* is the horizon, i.e. the max allowed number of steps
          to reach a state violating our invariant */
boolean Phi(state s);      state_probability_pairs next(state s);
Queue Q_old, Q_new;        Cache M;
double prob_Phi; /* incrementally stores the probability of
                    violating the invariant in at most k steps */
double max_prob_Phi; /* is the max allowed value for prob_Phi */
```

Fig. 3. Functions and Data Structures

we have just one initial state. FHP-Murφ keyword `invariant` defines the boolean function ϕ on S as well as the probability threshold β s.t. $P(\mathcal{S}, k, \phi) < \beta$ must hold (Remark 1).

The meaning of the declarations in Fig. 3 is as follows. Constant `k` (implementing k) is our verification horizon and is given to FHP-Murφ as a command line parameter. Functions `Phi()` implements ϕ. Function `next()` is the nextstate function of the PFSS $\mathcal{S}$ defined by FHP-Murφ input. Thus function `next()` takes a state s as argument and returns the set `next`(s) of pairs (t, p) s.t. s goes to t with probability p. Queues `Q_old` and `Q_new` are used to store distributions. Thus queue elements are pairs (s, p) where s is a state and p is the probability of reaching s from the initial state of $\mathcal{S}$. Such queues play, respectively, the same role as queues $Q(i)$ and $Q(i+1)$ in the while loop in Fig. 2. Queues `Q_old` and `Q_new` are the only place in which *state explosion* may occur in our algorithm. For this reason we implement them on disk analogously to [31]. This allows us to handle fairly large state spaces. The hash table `M` is a cache whose entries are pairs (s, p) as for queues `Q_old`, `Q_new`. Constant `max_prob_Phi` (implementing β) defines our probability threshold, i.e. the max allowed value for the probability `prob_Phi` of reaching (within the given horizon `k`) an error state (i.e. a state s s.t. `Phi(`s`)` $=$ `true`).

Note that from the above discussion follows that Murφ *hash compaction* (`-c`) [21] has no effect in FHP-Murφ since no FHP-Murφ data structure uses *state signatures* [29,30].

5.2 Functions `Search()` and `Insert()`

Our main function `Search()` is shown in Fig. 4. This function efficiently implements the computation described in Fig. 2.

Function `Insert()` is shown in Fig. 4. This function uses a cache table `M` in RAM to save queue space and thus computation time. `M[h]` returns the pair (s, p) stored in entry `h` of `M`. `M[h].state` denotes s and `M[h].prob` denotes p.

Every time it is necessary to enqueue a new pair (state s, probability p), `Insert(s, p)` is called. If state s is already stored in cache `M`, we simply update the stored probability in `M`, adding p to it. If state s is not stored in `M`, we check if the slot in `M` in which we have to put s is free. If it is free then we insert pair (s, p) in `M`. If it is not free, we call function `Checktable()` to empty `M` and then we insert pair (s, p) in `M`.

```
int Search() {
  prob_Phi = 0;
  enqueue(Q_old, (q, 1));  /* enqueue initial state q */
  for (level = 1; level <= k; level++) {
    clear cache table M;
    while (Q_old is not empty) {
      (s, p) = dequeue(Q_old);
      for all (s', a) in next(s) {
        if (Phi(s')) {
          prob_Phi = prob_Phi + p*a;
          if (prob_Phi >= max_prob_Phi)
            return(0); /* property does not hold */
        } else Insert(s', p*a);
      } /* for all */
    } /* while, level terminated, Q_old is empty  */
    Checktable();
    swap Q_new with Q_old ; /* now, Q_new is empty */
  } /* for */
  return(1); /* property holds */
}  /* Search() */

Insert(state s, double p) {
  if (s is in M) {
    h = hash(s);
    prob = M[h].prob + p;
    M[h] = (s, prob); /* new probability of s is prob */
  } else {
    collision = Insert_in_table(s, p);
    if (collision) {
      Checktable(); /* there is space to insert now */
      Insert_in_table(s, p);
    }
  }
} /* Insert() */

boolean Insert_in_table(state s, double p) {
  h = hash(s);
  if (M[h] is free) {
    M[h] = (s, p);
    return true;
  }
  else return (M[h].state == s);
} /* Insert_in_table() */

Checktable() {
  move M in Q_new and clear M; /* M is empty now */
}  /* Checktable() */
```

Fig. 4. Functions: `Search()`, `Insert()`, `Insert_in_table()`, `Checktable()`

If we were not using M, for each state s at level i we would have w copies of s in the queue, where w is the number of paths of length i leading to state s from initial state q. Using M rather than w copies of s we have just one or slightly more than one (depending on how large is M). This saves queue space as well as computation time. Hence, the more RAM available for M, the less our duplicated states, queue sizes, number of states to be explored and, finally, our computation time. For this reason M should be as large as possible.

5.3 Functions Insert_in_table() and Checktable()

Function Insert_in_table() is shown in Fig. 4. Function Insert_in_table() calculates the hash value h of s. If M[h] is a free slot, Insert_in_table() inserts s and p in M[h] and returns true. If M[h] is not free, Insert_in_table() returns false without inserting s and p in M.

Function Checktable() is shown in Fig. 4. It is the only function that enqueues values in Q_new; it simply flushes M into Q_new.

Function Checktable() is used by function Insert() to free M when a collision occurs. It is also called at the end of the while in function Search() (Fig. 4) to enqueue in Q_new the states visited after the last call to function Insert(), so that all states reached in the current level will be expanded in the next one.

6 Experimental Results

To show effectiveness of our approach we run two kind of experiments.

First, in Section 6.1, we compare FHP-Murφ with the probabilistic model checker PRISM [25].

Second, in Section 6.2, we run FHP-Murφ on a quite large probabilistic hybrid systems. Since our main goal is to use FHP-Murφ on hybrid systems, this second kind of evaluation is very interesting for us.

6.1 Probabilistic Dining Philosophers

In this Section we give our experimental results on using FHP-Murφ on the probabilistic protocols included in PRISM distribution [25]. We do not consider the protocols that lead to Markov Decision Processes or to Continuous Time Markov Chains, since FHP-Murφ cannot deal with them. Hence we only consider Pnueli-Zuck [23] and Lehmann-Rabin [20,26] probabilistic dining philosophers protocols. Moreover, we modify PRISM definitions for such protocols in order to have a finite horizon property to verify with FHP-Murφ. In fact, FHP-Murφ is unable to verify the PCTL properties for these protocols included in the PRISM distribution, since they are not of the required (*finite horizon probabilistic safety*) form $P_{<p}[\text{true } U^{\leq k}\phi]$.

Finally, FHP-Murφ definitions for such protocols have been obtained by translating into FHP-Murφ their PRISM (modified) definitions so that for each

protocol, FHP-Murφ and PRISM definitions specify exactly the same Markov Chain.

Our modifications to PRISM protocols consist in adding variables to count the number of times that a philosopher fails in getting both forks. We then verify that these counters are always less than a given maximum threshold (`MAX_CONT` in the following) with a given probability. This corresponds to verify *quality of service* properties, which are very frequent in practice. E.g., in the Pnueli-Zuck protocol, we changed the code fragment in Fig. 5 with the one in Fig. 6.

We want to know the probability $P(\texttt{MAX\_CONT}, k)$ of a counter reaching `MAX_CONT` in at most k (horizon) steps. We set $k = 20$ as our finite horizon (this value occurs in a property of the Lehmann-Rabin protocol in PRISM distribution [25]).

Fig. 7 shows the PCTL property to be verified stating that the probability that a counter reaches `MAX_CONT` has to be at most p. We set $p = 1$ since for computing $P(\texttt{MAX\_CONT}, k)$ the value of p does not matter.

In Fig. 8 we have the FHP-Murφ code corresponding to the PRISM code fragment of Fig. 6. Of course FHP-Murφ input language is the same as Murφ one [21], only FHP-Murφ has probabilities rather than booleans on rule guards. FHP-Murφ invariant `invariant p` γ requires that with probability at least p "all states reachable in at most k steps from the initial state satisfy γ" (k is FHP-Murφ horizon). Thus, using the notation in Section 5 we have that: $\phi = \neg\gamma$ and the probability threshold (`max_prob_Phi` in Fig. 3) is $(1 - \texttt{p})$.

Note that in Fig. 8 the probability threshold for FHP-Murφ invariant is 0, so that FHP-Murφ will not stop verification before completing all levels of the BF computation. This forces FHP-Murφ to compute $P(\texttt{MAX\_CONT}, k)$.

To assess FHP-Murφ effectiveness in Figs. 9, 10 we compare the results obtained with FHP-Murφ and with PRISM on, respectively, Pnueli-Zuck and Lehmann-Rabin protocols (modified as described above).

From Fig. 9 we can see that, for Pnueli-Zuck algorithm, when `NPHIL` $= 5$ (5 philosophers) and `MAX_CONT` is 4, PRISM is unable to complete any verification within 2GB of RAM, independently on which of the 3 PRISM verification algorithms (totally MTBDD based, algebraic and hybrid) is chosen. Similarly, for the Lehmann-Rabin algorithm, in Fig. 10 we see that when `NPHIL` is 4, and `MAX_WAIT` is 3, then PRISM is unable to complete the verification task in the same environment as above.

FHP-Murφ was always able to complete all given verifications tasks. Note however that, as it can be seen from Figs. 9 and 10, for the verifications tasks in which PRISM terminates, PRISM is always faster than FHP-Murφ.

Our experimental results show that for probabilistic protocols involving arithmetical computations FHP-Murφ is to be considered among the available (and valuable) tools for automatic finite horizon analysis of safety properties.

As for the numerical quality of FHP-Murφ we have that when both PRISM and FHP-Murφ terminate both give the same value for $P(\texttt{MAX\_CONT}, k)$ (column *Probability* in Figs. 9, 10).

```
module phil1
  p1: [0..10] init 0;
        .   .   .   .   .
  [] p1=6 -> (p1'=1);
  [] p1=7 -> (p1'=1);
        .   .   .   .   .
  [] p1=10 -> (p1'=0)
endmodule
```

Fig. 5. Pnueli-Zuck algorithm fragment to be modified in PRISM.

```
module phil1
  p1: [0..10] init 0;
  cont1: [0..3] init 0;
        .   .   .   .   .
  [] p1=6 & cont1!=MAX_CONT -> (p1'=1) & (cont1'=cont1+1);
  [] p1=6 & cont1=MAX_CONT -> (p1'=1);
  [] p1=7 & cont1!=MAX_CONT -> (p1'=1) & (cont1'=cont1+1);
  [] p1=7 & cont1=MAX_CONT -> (p1'=1);
        .   .   .   .   .
  [] p1=10 -> (p1'=0) & (cont1'=0);
endmodule
```

Fig. 6. Pnueli-Zuck algorithm modified fragment in PRISM.

```
P>=1.0 [true U<=20 ((cont1 = MAX_CONT) | (cont2 = MAX_CONT) |
                    (cont3 = MAX_CONT))]
```

Fig. 7. PCTL formula in PRISM.

```
function calc_prob(i : 1..NPHIL; c : 0..10) : prob;
-- probability that p[i] becomes c, NPHIL is the number of philosophers
begin
  switch p[i] -- p[1] corresponds to PRISM p1, p[2] to PRISM p2 etc
        .   .   .   .   .
    case 6: if (c = 1) then return 1.0 / NPHIL; else return 0.0; endif
    case 7: if (c = 1) then return 1.0 / NPHIL; else return 0.0; endif
        .   .   .   .   .
  endswitch;  end;

ruleset philosophers : 1..NPHIL do ruleset next : 0..10 do rule "next"
calc_prob(philosophers, next) ==> begin
  p[i] := c;
  -- cont[1] corresponds to PRISM cont1, cont[2] to PRISM cont2 etc
  if (c = 1 & (p[i] = 6 | p[i] = 7) & (cont[i] != MAX_CONT))
    then cont[i] := cont[i] + 1; endif;
  if (p[i] = 10 & c = 0) then cont[i] := 0; endif; end; end; end;

invariant "starvation" 0.0
forall i : 1..NPHIL do (cont[i] != MAX_CONT) endforall;
```

Fig. 8. Pnueli-Zuck algorithm in FHP-Murφ.

NPHIL	MAX_WAIT	Probability	Murφ memory	PRISM memory	Murφ time	PRISM time
3	3	7.335194164e-05	200	0.9057	51.970	1.487
3	4	6.883132778e-10	200	1.6844	52.610	2.507
4	3	1.88985976e-06	200	28.1066	242.940	28.72
4	4	2.910383046e-12	200	66.2659	244.170	71.112
5	3	9.164495139e-08	200	916.8246	1408.290	1023.468
5	4	4.194304e-14	200	N/A	1412.210	N/A
8	3	1.210429649e-10	1000	N/A	213790.740	N/A

Fig. 9. Results on a machine with 2 processors (both INTEL Pentium III 500Mhz) and 2GB of RAM. Murφ options: `-b` (bit compression), `-m200` (use exactly 200MB of RAM), `-maxl20` (the finite horizon is 20). The last verification had `-m1000` (use exactly 1GB of RAM). PRISM options: default options. N/A means that PRISM was unable to complete the verification; in this case, also the `-m` and `-s` (totally MTBDD and algrebraic verification algorithm respectively) have been used, with the same result. Memory occupations are in MB, time is in seconds.

NPHIL	MAX_WAIT	Probability	Murφ memory	PRISM memory	Murφ time	PRISM time
3	3	4.8039366e-06	800	39.0625	1040.330	84.556
3	4	0.	800	70.1483	1041.700	121.147
4	3	5.609882064e-08	800	N/A	34307.740	N/A

Fig. 10. W.r.t. Fig. 9, the only change is in the Murφ option `-m800` (use exactly 800MB of RAM).

6.2 Analysis of a Probabilistic Hybrid Systems with FHP-Murφ

In this section we show our experimental results on using FHP-Murφ for the analysis of a *real world* hybrid system. Namely, the *Control System* for the *Gas Turbine* of a 2MW *Electric Co-generative Power Plant* (ICARO) in operation at the ENEA Research Center of Casaccia (Italy).

Our control system (*Turbogas Control System*, TCS, in the following) is the heart of ICARO and is indeed the most critical subsystem in ICARO. Unfortunately TCS is also the largest ICARO subsystem, thus making the use of model checking for such hybrid system a challenge.

In [22] it is shown that by adding finite precision real numbers to Murφ, we can use Murφ to automatically verify TCS. In particular in [22] it has been shown the following. If the the speed of variation of the user demand for electric power (`MAX_D_U` in the following) is greater than or equal to 25 (kW/sec), TCS fails in maintaining ICARO parameters within the required safety ranges.

A TCS state in which one of ICARO parameters is outside its given safety range is of course considered an *error state*.

In [22] the user demand has been modeled rather roughly, using nondeterministic automata. Here we show that using FHP-Murφ we can define and, more importantly, automatically analyse, a more accurate model for the user demand by modeling it using a Markov Chain.

To do this we define a function $p(u, i)$ as follows:

$$p(u,i) = \begin{cases} 0.4 + \beta\frac{(u-M)|u-M|}{M^2} & \text{if } i = 1 \\ 0.2 & \text{if } i = 0 \\ 0.4 + \beta\frac{(M-u)|u-M|}{M^2} & \text{if } i = -1 \end{cases} \quad (1)$$

```
ruleset d_u : -1..1 do /* disturbance: takes values -1, 0 and 1 */
  rule "time step" user_demand(u, d_u) ==> main(u, d_u);
end; -- user demand disturbance
```

Fig. 11. Rulesets with probabilistic *user demand*

MAX_D_U	Reachable States	Rules Fired	Finite Horizon	CPU Time	Probability
25	3018970	8971839	1600	68562.570	7.373291768e-05
35	2226036	6602763	1400	50263.020	1.076644427e-04
45	1834684	5439327	1300	41403.150	9.957147381e-05
50	83189	246285	900	2212.360	3.984375e-03

Fig. 12. Results on a machine with 2 processors (both INTEL Pentium III 500Mhz) and 2GB of RAM. Murφ options used: `-b` (bit compression), `-m500` (use 500 MB of RAM). Time is given in seconds.

where $M =$MAX_U (maximum user demand value) and $\alpha =$MAX_D_U.

Denoting with $u(t)$ the user demand value at time t we can define the (stochastic) dynamics for the user demand as follows:

$$u(t+1) = \begin{cases} min(u(t)+\alpha, M) & \text{with probability } p(u(t),1) \\ u(t) & \text{with probability } p(u(t),0) \\ max(u(t)-\alpha, 0) & \text{with probability } p(u(t),-1) \end{cases} \quad (2)$$

In this way, we have that the further $u(t)$ from u_0, the higher the probability to return towards u_0, i.e. to decrement $u(t)$ if $u(t) > u_0$ and to increment it otherwise.

To see that (2) is indeed a Markov Chain, it is sufficient to observe that, $\forall\beta$, the sum of the outgoing transitions is obviously 1. Moreover, since $\frac{(u(t)-M)|u(t)-M|}{M^2} \leq 1$, as long as $-0.4 \leq \beta \leq 0.4$ holds, all probability values are between 0 and 1.

With FHP-Murφ the definition of Markov Chain (2), starting from the TCS model, is quite simple. This is done in Fig. 11, where `user_demand(u, d_u)` computes $p(u, d\_u)$ (1) and function `main` updates the system state, in particular updates `u` as described in (2).

In Fig. 12 we report the results of some verification runs done by FHP-Murφ with $\beta = 0.4$.

We are interested in cases where the error probability is greater than 0 (zero). From the results in [22] we know that this is the case if we choose MAX_D_U greater than or equal to 25 and the horizon value no smaller than the transition graph diameter. In our experiments here we choose our horizon as follows. Let Diam(n) be the diameter of TCS transition graph when MAX_D_U $= n$. We set our horizon k to be equal to $\lceil \frac{\text{Diam}(n)}{100} \rceil 100$. In this way we check the error probability in the error neighborhood.

Fig. 12 allows us to evaluate the probability of reaching an error state when MAX_D_U is greater than or equal to 25. Note that such a probability is rather

small, suggesting that in many cases setting `MAX_D_U` to 25 may be acceptable. This kind of evaluations are not possible with the nondeterministic verification of TCS carried out in [22].

7 Conclusions

We presented (Sections 3, 4) an *explicit* disk based verification algorithm for Probabilistic Systems defining *discrete time/finite state* Markov Chains. Given a Markov Chain and an integer k (horizon) our algorithm checks that the probability of reaching a given error state in at most k steps is below a given probability threshold.

We presented (Section 5) an implementation of our algorithm within a suitable extension of the Murφ verifier that we call FHP-Murφ (*Finite Horizon Probabilistic*-Murφ).

We presented (Section 6) experimental results comparing FHP-Murφ with (a finite horizon subset of) PRISM, a state-of-the-art symbolic model checker for Markov Chains. Our experimental results show that FHP-Murφ can handle systems that are out of reach for PRISM, namely those involving arithmetic operations on the state variables (e.g. hybrid systems).

Future work includes extending our approach to other models (e.g. Continuous Time Markov Chains) as well as to other kinds of PCTL formulas, e.g. formulas with unbounded until.

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